

A Real-Space Discrete Inverse Renormalization Group Method.

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Abstract. A numerical version of a real-space of the Inverse Renormalization Group (IRG) proposed in [1] is developed. It has been tested to obtain the scaling behavior of the random-forced heat equation in the short scales limit. Prospectives are described, and the most important target for the procedure is fully developed turbulence.

1. Introduction.

Fully developed turbulence stands as one of the most important unsolved problems in classical physics. The current description depends upon the qualitative picture developed by Richardson and its quantitative counterpart in Kolmogorov's work [2]. According to this view, fluctuations are settled at the flow in a typical length scale L due to a random stirring force or topographic irregularities. These fluctuations create vortices at the "integral scale" L , which develop other vortices of smaller size. The process stands producing lesser and lesser vortices until the viscous scale L_η is reached. Then the "fluctuating-energy cascade" is stopped and viscous dissipation takes place.

The main quantitative prediction of Kolmogorov theory is the value of the scaling exponents for the velocity correlators $\langle (v(x) - v(y))^n \rangle \approx |x - y|^{n/3}$. The experimental data are not completely in accordance with this prediction, just as mean-field theories are usually not completely in agreement with experiment [3]. A Renormalization Group (RG) analysis may enhance the results in a scale-invariant theory. Fully developed turbulence, according to Richardson-Kolmogorov picture is a scale-invariant theory, which suggests that a RG method might be implemented.

As it is argued in [7], turbulence and field theory are concerned with opposite limiting regimes. The integral scale is equivalent to a cutoff, but the physically interesting scales are *below* that cutoff scale. Reference [1] puts forward this argument in order to explain the failure of usual RG procedures applied to turbulence phenomena. The main problem is the appearance of an infinite number of expanding directions in the interaction parameters space. It is argued by Gawedzki et al. that a renormalization scheme fixing the integral scale (i.e.: the cutoff) and studying ever-decreasing scales would give a finite RG expansion. This procedure has been named **Inverse Renormalization Group**. The scaling behaviour of Kraichnan's passive scalar model [5] via IRG is performed in reference [1].

The motivation of this work is the development of a numerical implementation of a real-space IRG method, which we shall name Discrete Inverse Renormalization Group (DIRG).

2. The Real-Space Discrete IRG method.

The numerical procedure will be described in this section. Every numerical calculation about a partial differential equation involves a discretization of space and time. Our procedure adopts the common discretization of time ($t_n = n\Delta t$), but the space discretization is peculiar. Although we shall restrict ourselves to evolution (parabolic) equations, this restriction is not essential. Let us write the equation as

$$\partial_t u(t, x) = \mathcal{O}u(t, x)$$

where \mathcal{O} is any local operator, without any *a priori* retrictions.

The calculations shall be particularized to a 2-dimensional square domain with side length L and the topology of a torus (periodic boundary conditions), but we should remark that all these conditions might be easily removed. The square is divided by a coarse grain mesh into 3×3 little squares. (See figure 1.) The central little square is now divided into other 3×3 lesser squares, being the later grid 3 times finer. We may repeat the process up to any desired depth level. Let us call P the level. The procedure described so far shall be referred to as a Discrete IRG (DIRG).

The cells are indexed by three numbers: C_{ij}^n denotes the cell in the i^{th} column and j^{th} row of the n^{th} level. (See figure 2.) The indices run as follows: $i, j \in \{1 \dots 3\}$ and $n \in \{1 \dots P\}$. The DIRG description of the $u(t, x)$ field consists of the $3 \times 3 \times P$ values $u_{ij}^n(t)$ which represent the mean value of the field in the correspondent cell:

$$u_{ij}^n(t) = \int_{C_{ij}^n} dx u(t, x)$$

The $u_{ij}^n(t)$ must satisfy the consistency constraint: $u_{22}^n(t) = \frac{1}{9} \sum_{ij} u_{ij}^{n+1}(t)$ for all time. Initial data should be provided: the values of all the $u_{ij}^n(t_0)$ at a given time t_0 .

The following step is to set up DIRG algebraic equations by means of the suitable discretization of the partial derivatives equation. The essential point is the discretization of the derivative operators. Although there are other possible alternatives we have chosen an scheme which we have named “outward scheme”, that shall be exemplified with the explicit calculation of the action of ∂_x and ∂_y on $u_{12}^n(t)$:

$$\begin{aligned} (\partial_x u)_{12}^n(t) &= \frac{1}{2 \cdot 3^{n-1} L} [u_{13}^n(t) - u_{11}^n(t)] \\ (\partial_y u)_{12}^n(t) &= \frac{1}{3^{n-1} L} [u_{12}^n(t) - u_{12}^{n-1}(t)] \end{aligned}$$

Derivative operators are taken, when possible, by comparing with u -values in the same depth level, following the central scheme. When this is not possible, as in the ∂_y case in the former example, we compare the u -value at the actual cell with u -values in the *outer shell*, not in the inner one. This scheme is not the only one that is compatible with the DIRG scheme, because it privileges the relation of the evolution of the u -values at depth level n with those at outer depth levels, provoking an “inward information-flow”, which is characteristic of the Richardson-Kolmogorov view of turbulence.

3. The DIRG for the random-forced heat equation.

We have tested the DIRG method with an exactly soluble equation: the heat equation with a random source term:

$$\partial_t T(t, x) = \kappa \nabla^2 T(t, x) + f(t, x)$$

where $f(t, x)$ is a gaussian random process, with zero mean and covariance

$$\langle f(t, x) f(t', x') \rangle = \delta(t - t') C(x - x')$$

with $C(x - x')$ falling quickly to zero at large distances and nearly constant for $r \equiv |x - x'| < L$.

The statistics of the solution field may be obtained by a Martin-Siggia-Rose formalism or otherwise [1,4]. If we let $t_0 \rightarrow \infty$ then the mean value $\langle T(t, x) \rangle = 0$, and the two-point function depends on the covariance of the random force:

$$\langle T(t_1, x_1) T(t_2, x_2) \rangle = \int e^{|t_1 - t_2| \kappa k^2 - i k(x_1 - x_2)} \frac{\hat{C}(k)}{2 \kappa k^2} d\hat{k}$$

where $d\hat{k} = \frac{dk}{(2\pi)^d}$, as usually. Notice that the Fourier transform of the covariance of the random force is now taking the role of an ultraviolet cutoff.

Now we may try to obtain the asymptotic behaviour of this correlation function. We shall assume that the force acts only at large distances. In other words, if we let L be such a typical large length scale and $\Lambda = \frac{1}{L}$,

$$\hat{C}(k) = C \cdot \theta(|k| - \Lambda)$$

The correlation function now gives ($r = |x_2 - x_1|$, $\Delta t = t_2 - t_1$):

$$\frac{C}{\kappa r} \int_0^\Lambda dk \frac{\sin kr}{k} e^{\Delta t \kappa k^2}$$

The equal-time correlator becomes, making $x = r/L = \Lambda r$,

$$\langle T(x_1)T(x_2) \rangle = \frac{C}{\kappa r} \int_0^{\Lambda r} dx \frac{\sin x}{x}$$

For large distances it is easily seen that the correlation falls as r^{-1} , but the purpose of this calculation is to obtain the short-distance (compared to L) behaviour of the correlation function. The integrand should be approximated to second order in order to obtain:

$$\langle T(x_1)T(x_2) \rangle \approx \frac{C\Lambda}{\kappa} - \frac{C\Lambda^3}{12} r^2 + \mathcal{O}(r^4)$$

This means that, subtracted the constant, we obtain a scaling exponent 2, when looking at short distances. This exponent is also obtained in our DIRG approach. Let us explain how is this result achieved.

The DIRG analysis of this equation follows the steps of the preceding section. A random $f(t, x)$, decorrelated in time, is introduced only at the outmost level. All depth levels $n > 1$ evolve with a free heat equation, thus losing very early in the calculation all information referring to the initial condition. Fluctuations “cascade” inwards and induce a correlation function $T(t, x)T(t, x + \Delta x)$ which is time-averaged separately at each depth level.

Figure 3 plots the logarithm of this quantity against depth level. This is equivalent to a log-log graph of the correlation function with the space axis inverted. It is clearly seen that the resulting curve approaches asymptotically a straight line with slope 2.06 ± 0.09 . The analytical result, which gives a behaviour of $A + Br^2$, is thus satisfied by our approximate DIRG method.

This model is not completely suitable for an DIRG study due to the dimensionful constant κ , which prevents scale-invariance. As a consequence, it is difficult to reach very high depth levels. In our calculations we have only simulated 9 levels, but they are enough to appreciate the scaling behaviour (figure 3 shows only 8 levels because the last one is used in order to determine the other constant).

4. Conclusion and perspectives.

The main results expected in any DIRG analysis are scaling exponents of certain functions, specially correlation functions (type $\langle \prod_i^N T_i(t_i, x_i) \rangle$), structure functions (type $\langle (T(t, x) - T(t', x'))^n \rangle$). Generally speaking, all statistical parameters having scaling properties, or even multifractal behaviour, are possible candidates. Exotic scaling behaviours, such as discrete scale invariance [6], might profit from DIRG tools.

Notice that one of the advantages of the DIRG method over direct numerical simulation is that it is able to capture the scaling behaviour over various scale-decades saving a great deal of computer power. In order to simulate a given factor $S \equiv L/L_\eta$ usual numerical calculations require $\mathcal{O}(S^d)$, being d the space dimension, while DIRG needs only $\mathcal{O}(d \log S)$. The key is that DIRG focus on scaling behaviour, but it is not trying to obtain the whole dynamics.

The DIRG scheme described so far appears to be specially well suited for dealing with turbulence problems, according to the Richardson-Kolmogorov picture. The author is working on an implementation of the DIRG to Kraichnan’s passive-scalar problem. This is an exactly soluble model [5] in which we suppose certain time-decorrelated random velocity field and a contaminant (such as a solute or even temperature in the adiabatic limit) is advected through it. The exact solution shows a scaling universal behaviour with anomalous scaling exponents, i.e. not the ones given by naive dimensional analysis.

The main objective of these calculations are fully developed turbulence, in order to obtain numerically, following Gawedzki's ideas, the statistical properties of the flow. There is no point in remarking the importance for diverse branches of applied physics, such as astrophysics, geophysics, aeronautical engineering, meteorology... of a first-principles derivation of the scaling behaviour of the solution of any "turbulent equation": Navier-Stokes, Magneto-hydrodynamics...

Acknowledgements.

The author wishes to acknowledge Germán Sierra and Silvia N. Santalla for useful discussions. Silvia N. Santalla is also acknowledged for her collaboration in the preparation of the postscript figures.

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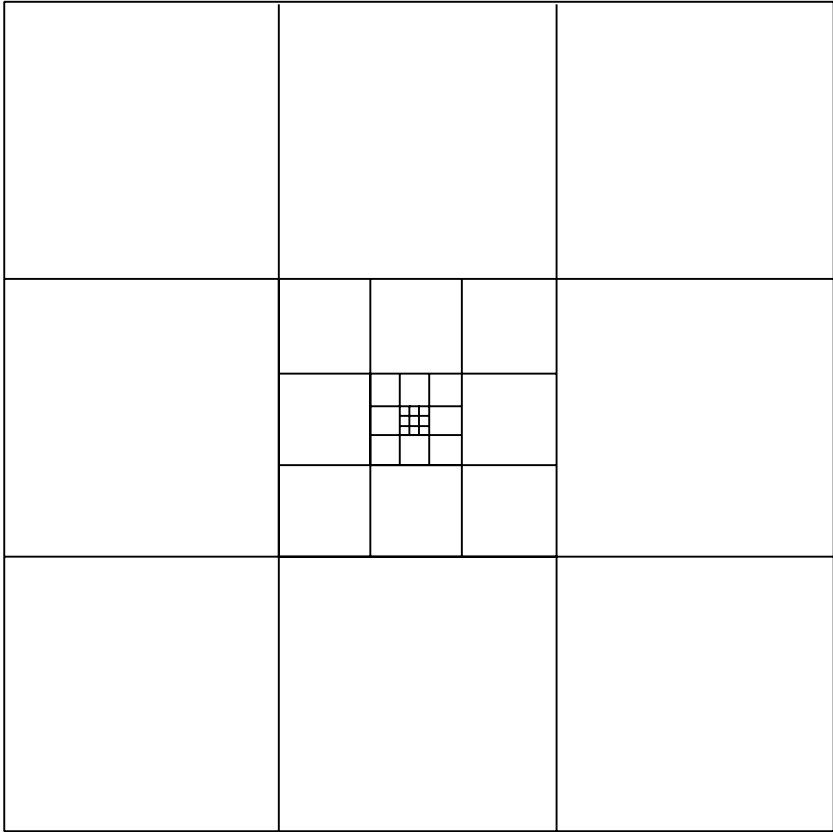
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Figure Captions.

Figure 1. The DIRG space discretization. The initial square domain is divided into a 3×3 mesh, which is itself divided into lesser and lesser squares, just as in Richardson-Kolmogorov picture.

Figure 2. The nomenclature of the cells in the DIRG space discretization. Notice also the lines between cells. Distances are taken along them in order to obtain the derivative operators.

Figure 3. DIRG results for the random forced heat equation. The vertical axis represents the logarithm of the two-point function and the horizontal axis is the depth level. A -2 slope straight line is superimposed to see the accordance for small scales (high depth level).



C_{11}^1	C_{21}^1		
C_{12}^1	C_{11}^2	C_{21}^2	C_{31}^2
	C_{12}^2		
	C_{13}^2		

